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Three-dimensional and multicellular steady and unsteady convection in fluid-saturated porous media at high Rayleigh numbers

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In an effort to determine the characteristics of the various types of convection that can occur in a fluid-saturated porous medium heated from below, a Galerkin approach is used to investigate three-dimensional convection in a cube and two-dimensional convection in a square cross-section. Strictly two-dimensional, single-cell flow in a square cross-section is steady for Rayleigh numbers R between $4\pi^2$ and a critical value which lies between 300 and 320; it is unsteady at higher values of R. Double-cell, two-dimensional flow in a square cross-section becomes unsteady when R exceeds a value between 650 and 700, and triple-cell motion is unsteady for R larger than a value between 800 and 1000. Considerable caution must be exercised in attributing physical reality to these flows. Strictly two-dimensional, steady, multicellular convection may not be realizable in a three-dimensional geometry because of instability to perturbations in the orthogonal dimension. For example, even though single-cell, two-dimensional convection in a square cross-section is steady at $R = 200$, it cannot exist in either an infinitely long square cylinder or in a cube. It could exist, however, in a cylinder whose length is smaller than 0·38 times the dimension of its square cross-section. Three-dimensional convection in a cube becomes unsteady when R exceeds a value between 300 and 320, similar to the unicellular two-dimensional flow in a square cross-section. Nusselt numbers Nu, generally accurate to 1 %, are given for the strictly two-dimensional flows up to $R = 1000$ and for three-dimensional convection in cubes up to $R = 500$. Single-cell, two-dimensional, steady convection in a square cross-section transports the most heat for $R < 97$; this mode of convection is also stable in square cylinders of arbitrary length including the cube for $R < 97$. Steady three-dimensional convection in cubes transports more heat for $97 \lesssim R \lesssim 300$ than do any of the realizable two-dimensional modes. At $R \gtrsim 300$ the unsteady modes of convection in both square cylinders and cubes involve wide variations in Nu.

1. Introduction

Finite-amplitude thermal convection in fluid-saturated porous material heated from below can take a variety of forms including both steady and unsteady, three-dimensional and multicellular motions. In this paper we discuss three-dimensional

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flows in a cube and strictly two-dimensional flows in a square cross-section. The circumstances in which these two-dimensional flows can be realized in a three-dimensional world are often quite limited because of instabilities which can occur through the other dimension.

On the basis of linear theory, steady two-dimensional convection with n cells in a square cross-section is possible for Rayleigh numbers R which exceed $\pi^2(n^2+1)^2/n^2$ (Horton & Rogers 1945; Lapwood 1948). Such flows have been studied, at Rayleigh numbers up to several times the critical value, by Elder (1967), Holst & Aziz (1972) and Palm, Weber & Kvernvold (1972). Here, we calculate steady two-dimensional convection for R up to 1000; Caltagirone (1975) has given results for R as large as 2000.

At sufficiently high Rayleigh number, two-dimensional convection becomes oscillatory (Horne & O'Sullivan 1974; Horne 1975, 1979; Caltagirone 1975). However, the Rayleigh number for the onset of this oscillatory behaviour has not been well determined. Horne & O'Sullivan (1974) investigated the Rayleigh number range 50 to 1250 and reported oscillatory behaviour only in the unicellular mode at $R \gtrsim 280$; their multicellular solutions in this range of Rayleigh number were steady. On the other hand, Caltagirone (1975) asserted that unicellular convection becomes oscillatory for Rayleigh numbers exceeding 384 ± 5 while bicellular convection becomes oscillatory for R between 800 and 1000. Based on these studies and results obtained in this paper, it appears that all two-dimensional multicellular patterns of convection become oscillatory with increasing Rayleigh number. The larger the number of cells is, the higher is the value of R at the onset of oscillatory behaviour.

One must exercise considerable care in attempting to apply the results of strictly two-dimensional studies to convection in three-dimensional geometries because the results simply may not carry over. Straus (1974) has shown that steady single-cell, two-dimensional convection cannot occur in infinitely long square cylinders for R larger than about 200, although the strictly two-dimensional calculations yield steady flows for R as large as about 300. The existence of steady, unicellular, two-dimensional convection in square cylinders with finite length can be determined from the stability analysis of Straus & Schubert (1978). If the ratio of the cylinder length to the dimension of the square cross-section lies between $0{\cdot}38n$ and $0{\cdot}61n$ (where n is any positive integer) then the flow cannot exist at $R > 200$. In particular, steady, unicellular, two-dimensional convection cannot exist in a cube at Rayleigh numbers larger than a value somewhat below 200. At $R = 200$, the flow can exist in stubby cylinders whose lengths are shorter than 0·38 times the dimension of the square cross-section. Similar limitations apply to the realization of steady two-dimensional convection with multiple cells in a three-dimensional geometry with square cross-section. Straus & Schubert (1978) have shown, at $R = 340$ and 400, that there are finite-length square cylinders in which no steady, two-dimensional convection can exist irrespective of the number of cells; at $R = 340$ this includes the cube. These values of R lie below and above the value $R \sim 380$, which was shows by Straus (1974) to be the largest R for which stable two-dimensional steady convection in an infinite layer is possible.

Steady three-dimensional convection in a cube can occur if $R > 4\pi^2$ (Beck 1972). Holst & Aziz (1972) have carried out numerical computations of this type of convection for $R = 60$ and 120, and Horne (1979) has obtained solutions for $R = 75$, 100 and 300.

Straus & Schubert (1979) have used the Galerkin technique to produce steady three-dimensional convection for Rayleigh numbers up to 150. When both steady two-dimensional and three-dimensional convection in a cube are possible, the mode adopted by the system depends solely on initial conditions (Straus & Schubert 1979). Heat transport is not necessarily maximized by the system. For $R \lesssim 97$, steady two-dimensional convection is realizable in a cube, and it transports more heat than steady three-dimensional flow. Zebib & Kassoy (1978) have investigated a three-dimensional flow which is maintained by the nonlinear interaction of orthogonal two-dimensional rolls. In a cube, this type of convection can occur for $4\pi^2 < R < 4{\cdot}5\pi^2$; it also transports less heat than does the two-dimensional flow. For $R > 4{\cdot}5\pi^2$, even though this type of flow may exist, we have found that three-dimensional convection in a cube is fundamentally different from the superposition of orthogonal two-dimensional rolls. Nevertheless, it does not transfer as much heat as does the unicellular two-dimensional roll until R exceeds 97. For $97 \lesssim R \lesssim 300$, steady three-dimensional convection in a cube is the form of motion that maximizes the Nusselt number Nu.

Based on the calculations reported in this paper and others carried out by Horne (1979), it can be concluded that three-dimensional convection in a cube becomes unsteady at high Rayleigh number. We have found time-dependent, three-dimensional convection for $R \gtrsim 320$. Either steady or unsteady three-dimensional convection occurs at a particular Rayleigh number, never both.

In summary, studies of two-dimensional convection need to be extended to determine more carefully the onset of oscillatory behaviour as a function of Rayleigh number and cell number. Attention must be focused on determining the realizability of two-dimensional flows in three-dimensional geometries. Investigations of three-dimensional convection need to be extended to higher Rayleigh number to determine the basic characteristics of the flows and their heat transport. We attempt to fill these needs by carefully calculating two-dimensional, multicellular, steady and unsteady convection in square cross-sections for R as high as 1000, and three-dimensional, steady and unsteady configurations in cubes for R up to 500. We use the Galerkin procedure described by Straus (1974) for two-dimensional convection and by Straus & Schubert (1979) for three-dimensional flows. To avoid repetition, we only provide some general information about the calculations and proceed immediately to a discussion of the results. The reader is referred to our previous papers for more detailed descriptions of the methods.

2. Notational and mathematical preliminaries

The Rayleigh number is

$$R = \frac{\alpha g \rho^2 K c d \Delta T}{\mu k}, \tag{1}$$

where α is the thermal expansivity of the fluid, g is the acceleration of gravity, ρ is the fluid density, K is the permeability of the porous medium, c is the fluid specific heat, d is the dimension of the square cross-section or the cubic box, μ is the fluid viscosity and k is the average thermal conductivity of the fluid and solid matrix. The Nusselt number Nu is the ratio of the horizontally averaged upward heat flux to that which would occur by conduction alone in the absence of convection.

N is a measure of the number of Fourier coefficients used in a calculation. Two-dimensional solutions are superpositions of modes whose horizontal (x direction) and vertical (z direction) structures are given by

$$\{\sin \text{ or } \cos(n\pi z/d)\}.\{\sin \text{ or } \cos(j\pi x/d)\}.$$

In the two-dimensional cases, the system was truncated according to $n+j \leqslant N$. Three-dimensional solutions have an additional sin or cos $(m\pi y/d)$ dependence (y is the other horizontal co-ordinate). In the three-dimensional cases the truncation criterion is $n+j+m \leqslant N$.

Our previous calculations for R as large as 150 (Straus & Schubert 1979), which were carried out without any *a priori* restrictions on the modal amplitudes (other than the truncation described above), always produced solutions with certain symmetry characteristics. In two dimensions, the only non-zero modes turned out to be those for which $\gamma n+j = \gamma k$, where γ is the number of cells ($\gamma = 1, 2, 3, \ldots$) and k is a non-zero even integer. In three dimensions, the non-zero modes satisfied $j+m$ and $n+m$ even. For the high Rayleigh number calculations reported in this paper, we used these symmetries, whose physical implications are discussed in Straus & Schubert (1979), to reduce the number of *a priori* unknown modal amplitudes. These symmetry requirements cannot be rigorously justified under completely general circumstances. We did test a number of solutions by numerically investigating whether the modes with zero amplitude in the symmetrized solutions would grow or decay from initial small amplitudes using a completely unrestricted calculation. The tests included both steady and unsteady, two- and three-dimensional solutions. In the steady cases, the modes which were not present in the symmetrized solutions decayed. However, in the unsteady cases these modes did not decay. Rather, they acquired fluctuating amplitudes comparable to many of the symmetrized modes. Although they contribute to the unsteady flow, they do not alter any of its essential characteristics such as the Nusselt number range, period, etc. They also do not appear to make any significant modification to the value of R at which solutions become unsteady. Table 1 summarizes how the number of non-zero modes used in both the symmetrized and non-symmetrized calculations depends on N. Our results show that the three-dimensional solutions, whether steady or unsteady, possess a symmetry in addition to those imposed, namely that the amplitudes of the n, m, j and n, j, m modes are identical. Thus the number of non-zero and independent modes in three dimensions is even smaller than that listed in table 1.

The two-dimensional flows are distinguished by the number of cells that would appear, for example, in the isotherm pattern as viewed in a vertical plane containing the motion. This appearance is determined by the dominant mode in the flow. Thus, for example, in single-cell convection $n = j = 1$ is the dominant mode, while in triple cell convection $n = 1, j = 3$ is the dominant mode. However, the $n = 1, j = 3$ mode can still contribute in a minor way to single-cell convection although the $n = 1, j = 1$ mode cannot contribute to triple-cell flow. The general character of a multicellular flow with γ cells is specified by the condition already noted, namely that $\gamma n+j = \gamma k$ (k is a non-zero even integer).

	N	Number of *a priori* non-zero modes	Number of non-zero modes after symmetry requirements
Two dimensions	8	36	20
	10	55	30
	12	78	42
	14	105	56
	16	136	72
Three dimensions	10	220	55
	12	364	91
	14	560	140

TABLE 1. The truncation scheme and the number of modes.

3. Discussion of results

(A) Steady solutions

Two- and three-dimensional convection is steady as long as the Rayleigh number is not too large. Tables 2 and 3 summarize the multicellular two-dimensional and three-dimensional cases, respectively. The change in Nusselt number with increasing N shows that the reported values of Nu (for the largest value of N) are generally accurate to within 1 per cent. The asterisks in table 2 indicate that for certain values of R, the accuracy of the reported Nu values were not quantitatively determined by carrying out calculations with $N > 16$. However, we believe that these Nusselt numbers are accurate to within a few per cent. The dependence of Nu on R is also shown in figure 1 for each of the multicellular two-dimensional and three-dimensional solutions.

Single-cell, two-dimensional convection is definitely steady for R as large as 300; two-cell convection is steady for R as large as 650 and three cells are steady for R up to 800. Three-dimensional convection is steady for R as large as 300. As discussed by Straus & Schubert (1979), the form of the convection is basically represented by the $n = j = m = 1$ mode. At Rayleigh numbers slightly larger than these values, the flows become unsteady. Single-cell, two-dimensional convection is definitely unsteady at $R = 320$, two-cell convection is unsteady at $R = 700$, and three-cell convection is unsteady at $R = 1000$. Three-dimensional convection is also unsteady at $R = 320$. The criterion we use to classify a flow as steady is that the time rate of change of each modal coefficient be at least three orders of magnitude smaller than the amplitude of the coefficient. When this criterion is satisfied, the computation is immediately terminated. The precise value of R at the onset of unsteady convection is difficult to determine numerically because the amplitude of oscillation tends to zero as the critical value of R is approached. We will discuss the time-dependent solutions in a separate section.

Single-cell, two-dimensional convection maximizes the heat transport in a cube for $R \lesssim 97$ (Straus & Schubert 1979), while three-dimensional convection has the largest Nusselt number for $97 \lesssim R \lesssim 300$. It is important to appreciate that the stability analyses of Straus (1974) and Straus & Schubert (1978) must be applied to ensure the realizability of the unicellular two-dimensional convection in the cube. The results of this paper by themselves would only support the conclusion that Nu for

Number of cells	R	Nu	N
1	39·48	1·0	—
	100	2·651	8
	150	3·320	8
	150	3·322	10
	200	3·808	10
	300	4·510	10
	300	4·514	12
2	61·68	1·0	—
	150	3·245	8
	200	3·986	8
	200	4·015	10
	200	4·022	12
	300	4·980	10
	300	5·005	12
	340	5·293	12
	400	5·660	12
	400	5·677	14
	600	6·599	14
	600	6·624	16
	650	6·813	16
3	109·66	1·0	—
	150	1·907	8
	200	3·130	10
	300	4·696	8
	300	4·876	10
	300	4·947	12
	340	5·449	12
	400	5·897	10
	400	6·045	12
	400	6·108	14
	600	7·421	14
	600	7·489	16
	800	8·312	16
4	178·27	1·0	—
	200	1·308	10
	300	3·438	10
	400	5·272	12
	600	7·573	14
	600	7·715	16
	800*	9·005	16
	1000*	9·846	16
5	266·87	1·0	—
	300	1·377	10
	400	3·304	12
	600	6·564	14
	600	6·593	16
	800*	8·602	16
	1000*	9·937	16

TABLE 2. Steady, two-dimensional calculations.

R	Nu	N
200	4·410	10
	4·497	12
250	4·999	10
	5·104	12
300	5·432	10
	5·590	12
	5·642	14

TABLE 3. Steady, three-dimensional calculations.

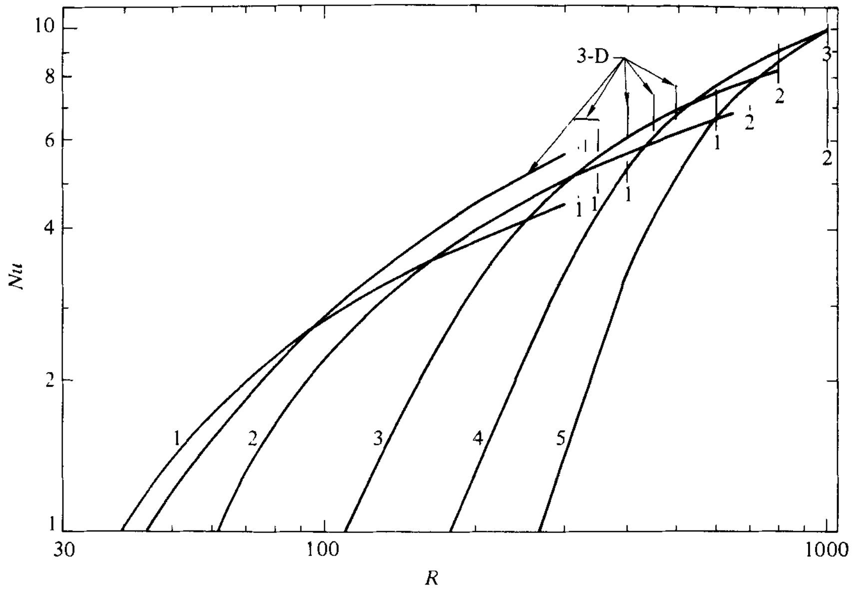

FIGURE 1. Nusselt number Nu *vs.* Rayleigh number for two- and three-dimensional convection. The integers refer to the number of cells in the two-dimensional flows. Convection is steady over the Rayleigh numbers for which the curves are shown. Vertical lines indicate the range of Nu observed in unsteady flows. At $R = 1000$, the Nusselt number for two-cell unsteady convection varies over the range indicated by the vertical line; also at $R = 1000$, Nu for three-cell unsteady convection varies over a range smaller than the size of the numeral 3 noted on the figure.

steady unicellular convection in a square cross-section exceeds Nu for steady three-dimensional convection in a cube for $R \gtrsim 97$. For $R \gtrsim 300$, the unsteadiness of both two- and three-dimensional convection makes it difficult to compare heat transports by different types of convection. Within the class of steady two-dimensional flows, single-cell convection has the largest Nusselt number for $R \lesssim 165$, double-cell convection has the maximum heat flow for $165 \lesssim R \lesssim 305$, triple cells maximize the heat transport for $305 \lesssim R \lesssim 535$, and four-cell convection has the highest Nu for

$$535 \lesssim R \lesssim 1000.$$

Recall that systems will not necessarily evolve to the states which maximize the heat transport. Instead, the final steady states chosen by a system depend solely on initial conditions (Straus & Schubert 1979; Horne 1979).

Our results for the heat transport by steady two-dimensional convection can be compared with a number of previously reported values of Nu. Caltagirone (1975) gives values of Nu for single cells at $R = 100$, 200 and 300 which are identical to the Nusselt numbers in table 2. He also gives Nu for two-cell convection at $R = 200$ and 300 in exact agreement with our calculations. His value of $Nu = 6{\cdot}822$ for four cells at $R = 500$ compares well with our value of 6·75 from figure 1. At $R = 1000$ he gives $Nu = 10{\cdot}190$ for four cells, some 3·5 % larger than our value of 9·846. For five-cell, two-dimensional convection we can compare with his Nusselt numbers at $R = 500$, 800 and 1000. His values are 5·34, 9·15, and 10·63, respectively. Our values are 5·1 (from figure 1), 8·602 and 9·937, respectively. Caltagirone's (1975) Nusselt numbers are high by about 4·7 %, 6·4 %, and 7 %, respectively. Caltagirone's (1975) finite-difference computations overestimate Nu by larger percentages as R increases. This is most likely a consequence of the fact that he did not refine his grid size with increasing R; instead he used a constant 48×48 grid for all Rayleigh numbers greater than 200. We have no way of assessing the accuracy of the solutions he reports at $R = 2000$. Horne's (1975) values of Nu for one-, two-, and three-cell convection at $R = 250$, 375 and 500, respectively are 4·51, 6·17, and 7·79, substantially larger than our determinations of 4·18, 5·5, and 6·9. Once again, we would attribute the overestimation of Nu to too coarse a grid in the finite-difference calculations. Horne's (1979) value of $Nu = 2{\cdot}8$ for single-cell convection is already 6 % too high at $R = 100$. Finally, Holst & Aziz (1972) gave $Nu = 3{\cdot}49$ for single-cell convection at $R = 120$, a considerable overestimate (figure 1).

Some Nusselt number comparisons are also possible for steady three-dimensional convection. Horne's (1979) Nusselt numbers for $R = 75$ and 100 are only slight overestimates, while his value of $Nu = 6{\cdot}45$ for $R = 300$ is rather high compared with our calculation of $Nu = 5{\cdot}61$. Holst & Aziz (1972) found $Nu = 3{\cdot}94$ at $R = 120$ and 1·67 at $R = 60$, compared with the accurate values of $Nu = 3{\cdot}16$ and 1·55. The analytic formula given by Zebib & Kassoy (1978) for the Nusselt number of a three-dimensional flow is inapplicable to convection in a cube with $R > 4{\cdot}5\pi^2$.

The onset of oscillatory convection in two dimensions has been reported by Caltagirone (1975) to occur at $R = 384 \pm 5$ for single cells; we find the onset at a value of R between 300 and 320. For two cells, Caltagirone (1975) reports that oscillatory behaviour sets in at R between 800 and 1000; we find the occurrence between $R = 650$ and 700. Horne & O'Sullivan (1974) state that unicellular flow is oscillatory at $R \gtrsim 280$; Horne (1979) finds oscillatory single-cell convection at $R = 300$. These results are in approximate agreement with our conclusions.

(B) *Unsteady solutions*

We have already discussed the fact that both the two-dimensional, multicellular solutions and the three-dimensional solutions become unsteady for sufficiently large R. We have also given the approximate values of R at which the unsteadiness sets in for the different types of convection. Although we have not carried our computations to large enough R to observe unsteadiness in two-dimensional solutions with four or more cells, it is reasonable to conclude that all multicellular solutions will eventually become unsteady as R increases. The critical values of R at which oscillatory two-dimensional convection sets in are approximately proportional to the number of cells, at least for one-, two- and three-cell convection. What is surprising, and disappointing in a sense, is that three-dimensional convection also becomes unsteady and at a

Number of cells	R	Nu	N	Period
1	320	4·62–4·63	10	0·014
	350	4·78–5·13	10	0·013
	400	5·09–5·45	12	0·012
	600	6·3–7·6	14	0·005–0·008
2	700	6·93–7·07	16	0·005–0·007
	800	7·8–9·3	16	0·010
	1000	5·8–11·3	16	0·013
3	1000	8·87–8·93	16	0·004

TABLE 4. Oscillatory, two-dimensional calculations.

R	Nu	N	Period
320	5·75858–5·75865	12	0·004
330	5·714–5·978	12	0·004
350	5·7–6·3	12	0·004
400	6·1–7·0	14	0·0045–0·0065
450	6·3–7·4	14	0·003–0·006
500	6·6–7·7	14	0·002–0·006

TABLE 5. Unsteady, three-dimensional calculations.

Rayleigh number approximately the same as the one at which unicellular two-dimensional convection becomes unsteady. Horne (1979) has also found unsteady three-dimensional solutions at $R = 300$ and 400. The disappointment arises from the fact that, since the forms of convection are unsteady at high R, it is difficult to imagine how high Rayleigh number solutions could ever be produced without carrying out direct numerical computations. If convection were steady at high R, approximate descriptions of the solution, for example by boundary-layer theory, might be possible. Of course, it is conceivable that approximations would be found to adequately characterize aspects of unsteady flow; we are simply asserting that it will be much more difficult to come by these approximations for time-dependent flows.

Tables 4 and 5 list the two- and three-dimensional flows we have found to be unsteady. Figures 2 and 3 show how the Nusselt number fluctuates with time for a few of these cases. It is clear from these figures that the fluctuations are generally complex in character and not repeatable in detail, making it difficult to precisely characterize these flows. Thus there is some uncertainty in the tabular entries. The ranges of Nu listed in the tables represent the smallest minimum and the largest maximum values of Nu observed during the extent of the computation. These values could change had we carried out the calculations for longer times (hopefully by not very much, however). These Nusselt number ranges are shown on figure 1 by the vertical lines. The periods in tables 4 and 5 represent typical times between succeeding maxima of Nu. (When a range, rather than a single value is given for the period, it indicates that we observed the period to vary over that range.) In all cases, the time step used in the calculations was at least an order of magnitude smaller than the period of oscillation.

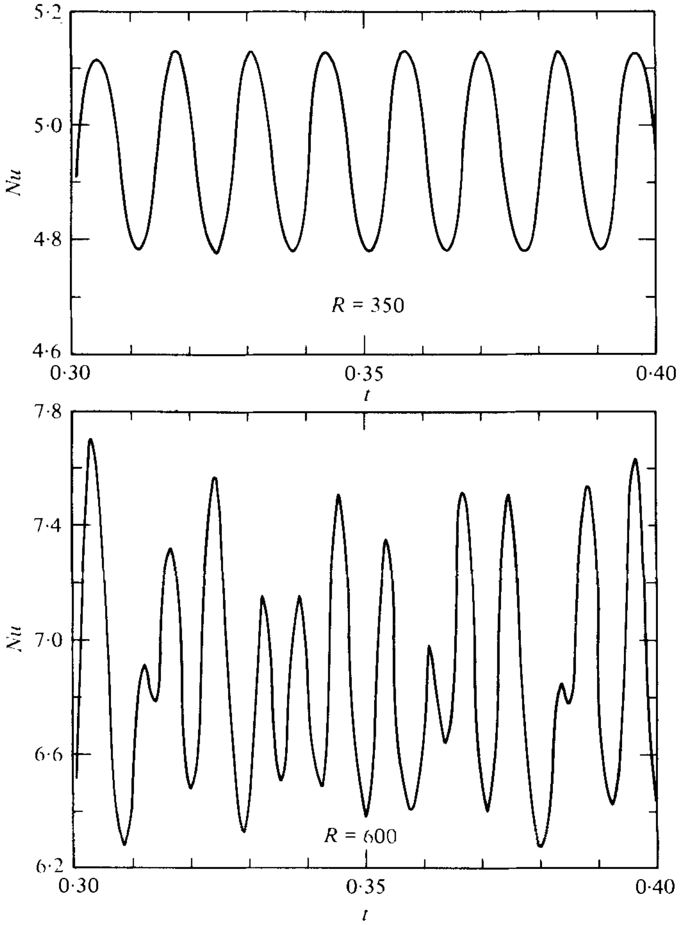

FIGURE 2. Nusselt number variations with time t in unsteady, two-dimensional, unicellular flows at two values of R. The time scale is in units of the thermal diffusion time across the square.

For a given type of convection, the amplitude of the Nusselt number fluctuation generally increases with R. When three-dimensional convection in a cube is steady and for $R \gtrsim 97$, it transports the maximum amount of heat. When three-dimensional convection in a cube is unsteady it transports more heat at the maxima of its Nu fluctuations than do any of the steady or unsteady two-dimensional flows in square cross-sections. However, the heat transport at the minima of the Nu fluctuations can fall below that of the two-dimensional solutions. Similar observations can be made about the heat transported by the multicellular two-dimensional solutions. For example, the heat flux carried by steady double-cell convection in a square exceeds that of steady unicellular convection for $R \gtrsim 165$. However, when the unicellular pattern becomes unsteady, it can transfer more heat at the maxima of the Nu fluctuations than does the still steady double-cell flow. For example, this can be seen in figure 1 at $R = 600$, where fluctuating unicellular convection has a higher maximum Nu than even the steady triple-cell flow. Even the average Nusselt number of one-cell flow exceeds Nu of the steady double cell at $R = 600$. Note that the maximum Nu of unsteady double-cell convection exceeds Nu of even the five-cell steady flow at

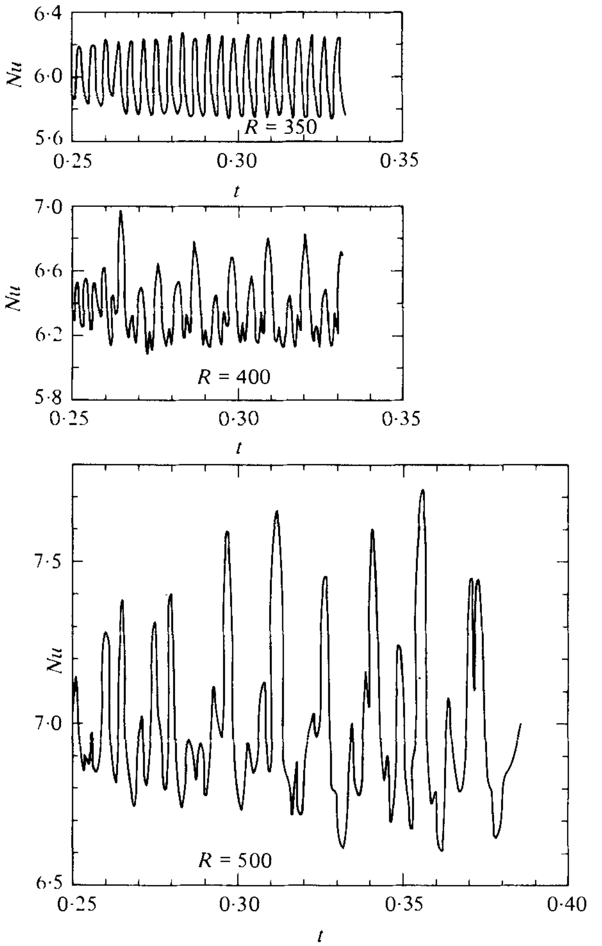

FIGURE 3. Nu as a function of t in unsteady three-dimensional convection.

$R = 1000$. When the flows become unsteady, the average values of Nu do not define a smooth, natural extension of the steady Nu *vs.* R curves.

The character of the unsteadiness in two dimensions is shown in figure 2 for single-cell flows at $R = 350$ and 600. At the lower Rayleigh number, the oscillation in Nu is a simple, essentially repetitive fluctuation between two values with a well-defined period. At $R = 600$, the situation is quite different. There are both high and low maxima (and minima as well), no repeatable pattern can be discerned, and the time interval between successive maxima shows a clear variation which cannot be attributed to the time resolution of the computations. (The periods are at least an order of magnitude longer than the time steps used.) From table 4, it can be seen that the period tends to decrease with R for the single-cell flows. However, the power-law dependence of period on R suggested by Horne & O'Sullivan (1978) seems to be an oversimplified description of the unsteadiness. In this connexion, table 4 also shows an increase in period with R for double-cell convection.

Figure 3 and table 5 describe the character of unsteady three-dimensional flows. The periods of the Nusselt number fluctuations are about a factor of 3 smaller than those observed for unsteady single-cell convection at comparable Rayleigh numbers. The period is essentially independent of R in the range of Rayleigh number considered. The fluctuations are relatively simple at $R = 350$; except for period and amplitude, they resemble those of the unicellular oscillatory flow shown in figure 2 for $R = 350$. At $R = 400$, the variations in Nu with time are more complex. There are both high and low maxima and period variations not unlike those encountered in the high Rayleigh number two-dimensional case of figure 2. There is a qualitative repeatability to the $R = 400$ case of figure 3 in that the high and low maxima occur alternately. At $R = 500$, the Nu *vs.* time pattern is still more complex with larger period variations and less repeatability.

Laboratory experiments dealing with the onset of fluctuating convection in porous media have been carried out by Combarnous & LeFur (1969), Caltagirone, Cloupeau & Combarnous (1971) and Horne & O'Sullivan (1974). In particular, Caltagirone *et al.* (1971) carried out experiments in porous layers with two different horizontal cross-sections, one nearly square and the other highly elongated. Fluctuating three-dimensional convection set in at Rayleigh numbers between 240 and 390 in the nearly square geometry; in the elongated geometry, two-dimensional unsteady convection set in at values of R between 190 and 305. This behaviour is consistent with our finding that the transition to fluctuating convection occurs at a value of R between 300 and 320 for both two- and three-dimensional flows.

4. Realizability of two-dimensional flow in three-dimensional configurations

In order to keep computational time to a minimum, the two-dimensional solutions of this paper were calculated using an algorithm which did not admit motions in the third dimension. Thus, these strictly two-dimensional flows may not be realizable in a three-dimensional geometry, as has already been noted. Each of the two-dimensional flows could be used to initialize a fully three-dimensional computation as a way of determining the stability of the two-dimensional flow to perturbations in the third dimension. However, such a procedure at least partially offsets the original gain in computational time. Alternatively, one could use the stability analyses of Straus (1974) and Straus & Schubert (1978) to investigate the existence of the two-dimensional solutions in three-dimensional geometries. Straus's (1974) result can be used to test the stability of two-dimensional flows in square cross-sections to perturbations in the other dimension only in infinitely long cylinders with square cross-section. The results of Straus & Schubert (1978) can test the stability in cylinders of finite length, however only at a few values of Rayleigh number. Although a particular two-dimensional solution may not exist in the infinitely long cylinder, it may nevertheless exist in the finite cylinder because the perturbation which destabilizes it in the infinite cylinder may not fit within the cylinder of finite length.

First, let us summarize what we can learn from Straus (1974) relevant to the realizability of the two-dimensional flows. He has shown that steady two-dimensional convection cannot exist in infinitely long cylinders with square cross-section at R greater than ~ 200 and ~ 322 for flows with one and two cells, respectively. Convection with two cells is also unstable for R between $\frac{25}{4}\pi^2$ and 140. Multicellular convection with three or more cells is unstable for all R. Since the strictly two-dimensional calculations of this paper show that steady flows with one, two and three cells occur for R as large as 300, 650 and 800, respectively, and steady motions with four and five cells occur for R very much in excess of 1000, there are large ranges of Rayleigh number in which the steady, multicellular, two-dimensional flows simply do not exist in the infinitely long cylinder. The transitions from steady to oscillatory two-dimensional convection are therefore meaningless for this geometry.

From Straus & Schubert (1978), we can see how the finite length of a square cylinder modifies these conclusions, at least at a few values of R. At $R = 100$, single-cell flows are stable for any cylinder length. Two-cell flows are stable to rolls in the orthogonal direction for cylinders with length smaller than 0·56 times the dimension of the square cross-section. Although double-cell convection cannot exist in the infinitely long cylinder at $R = 100$, it can exist in sufficiently short cylinders. At $R = 200$, single-cell flows are stable in square cylinders shorter than 0·38 times the dimension of their cross-section. Unicellular flow is unstable if the cylinder length lies between $0{\cdot}38n$ and $0{\cdot}61n$ (n is any positive integer) times the cross-section dimension; this includes the cube. Double-cell convection is stable at $R = 200$ consistent with the results of Straus (1974). A two-dimensional flow which is stable in the infinitely long cylinder cannot be unstable in the finite length cylinder because all possible orthogonal perturbations can exist in the infinite cylinder. For a Rayleigh number of 340, Straus & Schubert (1978) show that no steady multicellular solutions are possible in a cube or in cylinders with lengths between $\sim 0{\cdot}49$ and $\sim 0{\cdot}78$, and between $\sim 0{\cdot}35$ and $\sim 0{\cdot}39$ times the cross-section dimension. This conclusion even applies to two-dimensional convection in the orthogonal direction. Any multicellular flow can exist if the cylinder is extremely stubby, i.e. if its length is smaller than $\sim 0{\cdot}18$ times the cross-section dimension. This is because no orthogonal mode can exist in this case according to linear stability theory. Double-cell flow can even be found in a cylinder whose length is smaller than $\sim 0{\cdot}21$ times its cross-section dimension. Straus & Schubert (1978) also present results for $R = 400$. The persevering reader can deduce the limitations on the realizability of two-dimensional flows at this Rayleigh number.

This discussion is by no means exhaustive of the implications of the stability analyses for the existence of the two-dimensional solutions. It should indicate the complexity of the issue and emphasize, as already mentioned, the extreme caution that must be exercised in comparing two- and three-dimensional solutions, and attributing physical reality to the two-dimensional ones.

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